The Role of Nonemotional Cognitive Feelings in Constructing Proofs

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Abstract. We describe a perspective and a framework for understanding the role of nonemotional cognitive feelings in proving theorems. We begin with a brief discussion of the nature of affect, emotions, and nonemotional cognitive feelings. We see kinds of situations as mentally linked to kinds of feelings that then participate in enacting behavioral schemas to yield actions and introduce the idea of a < situation, action > pair. We see certain feelings and some parts of procedural knowledge as driving certain aspects of proving. The genesis of the < situation, action > link and the associated feeling is illustrated by describing how, for one student, a feeling of rightness/appropriateness became linked to a specific act of proving.

1. Introduction

It is our intent to broaden the discussion of affect in the mathematics education research literature to include nonemotional cognitive feelings (Clore, 1992). We describe a perspective, and an accompanying theoretical framework, for understanding and studying the role of nonemotional feelings in cognition, specifically in proving theorems. To do this, we consider how certain small mathematical actions depend on alternating conscious and nonconscious mental states, and note that some actions invoked during proof construction can involve procedural knowledge of a very small grain-size – something that the first two authors have called *behavioral schemas* (Selden & Selden, 2008). In doing so, we call on perspectives from mathematics education, neurology, psychology, and philosophy.

2. Related literature

2.1 Affect

McLeod (1992) has considered three aspects of affect -- beliefs, attitudes, and emotions -- and described them as increasing in intensity and decreasing in stability with emotions being the most intense and the least stable. To McLeod's theoretical framework, DeBellis and Goldin (1997, 1999, 2006) have added a fourth aspect, values. For them, affect "is not auxiliary to cognition; it is centrally intertwined with it" (Hannula, et al., 2004). They see "affect as a highly structured system that encodes information, interacting fundamentally – and reciprocally – with cognition" (1999, p. 2-249). They also note as do Ortony, Clore, and Collins (1988) that feelings have an appraisal value that can either be positive or negative. In addition, Clore (1992) has considered both affect-as-information and feelings-as-information.

Within the affective literature on cognition, there is no consensus on what consititutes a "feeling." DeBellis and Goldin (2006) refer to "emotional feelings." Damasio (2003), however, distinguishes emotions from feelings, saying that the former are public and the latter private (p. 27). "Emotions play out in the theater of the body. Feelings play out in the theater of the mind" (p. 28). Most of the emotions that Damasio considers, such as joy and sorrow, as well as some less intense emotions, are a complex collection of chemical and neural responses to a stimulus (p. 53) that may produce bodily changes, such as changes in one's heart rate, temperature, and so forth. Such emotions often give rise to feelings. We regard the emotional feelings of DeBellis and Goldin as feelings which have associated emotions.

2.2 Feelings

Clore (1992) distinguishes emotional feelings from nonemotional cognitive feelings, such as a feeling of knowing. For example, one might experience a feeling (of knowing) that one has seen a useful theorem before, but not be able to bring it to mind immediately. Such feelings of knowing can guide cognitive actions because they can influence whether one continues a search or aborts it (Clore, p. 151).

DeBellis and Goldin speak of local and global affect and of affective pathways that may be established after repeatedly experiencing "sequences of (local) states of feeling" that are positive or negative but rarely, if ever, neutral. For example, in problem solving, they explain that a situation that begins with curiosity and bewilderment can lead to frustration. This, in turn, can invoke a positive affective pathway that leads to a change in strategic thinking, which if successful, is followed by pleasure, elation, and satisfaction. Or, such frustration can lead to invoking a negative affective pathway that, when known procedures fail, may lead to anxiety and despair (DeBellis & Goldin, 2006, p. 134).

Their notion of affective pathways is that repeated experiences with certain problem solving situations may come to consistently evoke the same emotion, and that emotion will consistently affect behavior in a certain way. For example, a beginning problem solver might consistently feel frustration after a number of initial problem-solving attempts are unsuccessful. This feeling of frustration may then consistently lead a student to give up on problem-solving tasks, ask the teacher for hints, or something else.

2.2.1 Nonemotional cognitive feelings

Some nonemotional cognitive feelings, different from a feeling of knowing, are a feeling of familiarity and a feeling of rightness. Mangan (2001) distinguishes the two.

Of the former, he says that the “intensity with which we feel familiarity indicates how often a content now in consciousness has been encountered before,” and this feeling is different from a feeling of rightness (Mangan, 2001, Section 1, Paragraph 3). It is rightness, not familiarity that is “the feeling-of-knowing in implicit cognition” (Mangan, 2001, abstract). Rightness is “the core feeling of positive evaluation, of coherence, of meaningfulness, of knowledge” (Mangan 2002, Section 1, Paragraph 11).

Feelings are sometimes separated from emotions in such a way that the feelings themselves have no sensory component, but rather are non-sensory experiences. Because they do not involve the senses, non-sensory experiences are, for example, not red or hot. Also, they are not usually expressible in words. “The feeling of familiarity is not a color, not an aroma, not a taste, not a sound. It is possible for the feeling of familiarity to merge with, or be absent from, virtually any sensory content found on any sensory dimension” (Mangan, 2001, Section 1, Paragraph 7). From the point of view of problem solving or proving, an important non-sensory experience is a feeling of rightness (James, 1890; Mangan, 2001).

In regard to a feeling of rightness, Mangan (2001, Section 6, Paragraph 7) says “people are often unable to identify the precise phenomenological basis for their judgments, even though they can make these judgments with consistency and, often, with conviction. To explain this capacity, people talk about ‘gut feelings,’ ‘just knowing,’ hunches, intuitions.”

In the following, we focus on problems, or perceived tasks, of a very small grain-size and on feelings that are often not intense, such as feelings of rightness, familiarity, appropriateness, comfort, or caution. Such feelings are non-sensory experiences that, at any particular moment, can pervade one's whole conscious field. They may be rather "vague" and not easily noticed or focused upon, but can influence one's actions (Mangan, 2001, Section 1, Paragraph 4).

3. Feelings of rightness

Feelings of rightness can give direction, be summative, or suggest appropriateness. Of a feeling of rightness/direction, Mangan says, "In trying to solve, say, a demanding math problem, [a feeling of] rightness/wrongness gives us a sense of more or less promising directions long before we have the actual solution in hand" (2001, Section 6, Paragraph 3). A feeling of rightness/summation can integrate "large sets of information necessary for the problem-solving [or theorem proving] processes" (Damascio, 2001, p. 177).

It appears that at the end of a proof validation, short-term memory is inadequate to hold sufficient information to allow a rational judgment of whether the proof is correct. However, something must cause an individual to decide his own, or someone else's, proof is correct. We see a (summative) feeling of rightness as playing a major role in such decisions.

3.1 Procedural knowledge, situations, and actions in proving

We now focus on problems, or perceived tasks, of a very small grain-size in constructing or validating proofs. Some of these tasks can be viewed as procedural, and for experts, can become automated. By procedural, we mean *knowing how*, as opposed to *knowing that or why*. Because of the small grain-size and the nature of proof construction, such procedural knowledge should not necessarily be regarded as a procedure, or a part of a procedure, such as an algorithm for long division.

Situations that occur in constructing or validating proofs can be paired with actions through the enactment of behavioral schemas (Selden & Selden, 2008). Sometimes a < situation, action > pair is lastingly mentally linked to a specific kind of feeling, such as a feeling of appropriateness; that is, the action is seen as being appropriate to the situation. In that case, the behavioral schema linking the situation to the action may be strengthened. Also, feelings of inappropriateness can correspondingly influence behavioral schemas. The links and the associated behavioral schemas are part of an individual's knowledge base, and the enactment of behavioral schemas is part of an individual's cognition.

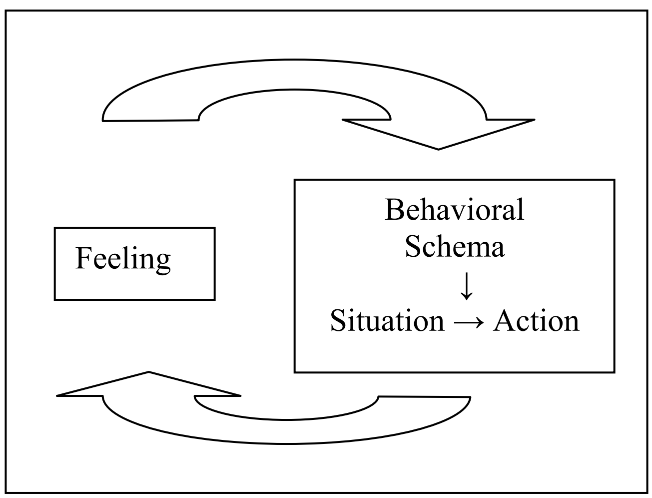

Figure 1

An individual's perception of a situation may trigger the enactment of a behavioral schema to produce an action that results in an associated feeling. On the other hand, an individual's perception of a situation may trigger this same feeling which, in turn, influences the enactment of the behavioral schema, and thereby strengthens, or weakens, the behavioral schema associated with the < situation, action > pair. See Figure 1.

In particular, certain feelings and some parts of procedural knowledge can be seen as driving certain aspects of proving. For example, a prover might recognize or notice that the statement to be proved starts with mixed universal and existential quantifiers, as often happens in beginning real analysis theorems. Specifically, the situation of proving "For all $\varepsilon > 0$" may be associated with the action of writing "Let ε be a real number greater than 0," meaning ε is arbitrary, but fixed. Such an association may be lastingly linked with a feeling, such as a feeling of rightness/appropriateness.

4. The data

The events we describe have three parts: (1) the paring of a situation and an action; (2) the understanding of a warrant for why the action should be carried out in the situation; and (3) a feeling of rightness/appropriateness associated with the < situation, action > pair and its warrant. From our background in mathematics education, we would expect that a feeling of rightness/appropriateness might arise more or less simultaneously with an understanding of the warrant for the < situation, action > pair, that is, why the action should be carried out. However, our expectation has *not* been realized in two different instances of student work.

4.1 Recognizing a situation, knowing the warrant for the action, but not carrying it out

In our recent teaching, we have encountered several students, one of whom will be described below, that seemed to recognize a situation, were aware of the associated action, and indicated on numerous occasions that they properly understood the warrant for the action. However, they failed to carry out the action. We suggest these students did not have a feeling of rightness/appropriateness for the action.

The setting was a design experiment, consisting of a Modified Moore Method course[1], whose sole purpose was to improve the proving skills of beginning graduate and advanced undergraduate mathematics students. The course was consistent with a constructivist point of view, in that we attempted to maximize students' proof writing experiences. It was also somewhat Vygotskian in that we represented to the students how the mathematics community writes proofs.

The students were given self-contained notes consisting of statements of theorems, definitions, and requests for examples, but no proofs. The students presented their proofs in class, and the proofs were critiqued. Suggestions for improvements in their notation and style of writing were also given. There were no prepared lectures, and all comments and conversations were based on the students' work. It was a three-credit one-semester course that met for an hour and fifteen minutes twice a week, making 30 class meetings altogether. The course covered some basic ideas about sets, functions, real analysis, and semigroups. The specific topics covered were of less importance than giving students opportunities to experience as many different types of proofs as possible.

4.1.1 The case of Edward

On Day 16, just over halfway through the course, Edward was proving that the identity function on the set of real numbers is continuous. We were using the following definition: A function f is *continuous* at a means for all $\varepsilon>0$, there is a $\delta>0$ so that for all x, if $|x-a|<\delta$ then $|f(x)-f(a)|<\varepsilon$.

[1] For more information on the Moore Method, see Jones (1977) or Mahavier (1999).

In order to write a proof, Edward needed to link “For all ε>0” in the definition of continuity [the situation] to writing in the proof something like “Let ε be a number greater than 0”, meaning ε is arbitrary, but fixed [the action]. Edward was aware of the warrant for this action, namely, that because ε was arbitrary, anything proved about it must be true for every ε. At least eleven times previously in class, Edward had seen this kind of < situation, action > pair, including its warrant. For example, on Day 7, Edward had proved $A \setminus (B \bigcup C) = (A \setminus B) \bigcap (A \setminus C)$ essentially correctly, by choosing an element *x* in one side and proving *x* is an element in the other side, and vice-versa.

Here is Edward’s proof that the identity function is continuous on the set of real numbers. Let $a \in R$ and $\varepsilon > 0$ and $\delta > 0$ where $\delta = \varepsilon / 2$. So $\delta < \varepsilon$. For any $\varepsilon > 0$ and any $x \in R$ such that $|x - a| < \delta$. We have that $|f(x) - f(a)| < \delta < \varepsilon$ therefore f is continuous.

In the critique that followed, the teacher said it was inappropriate to write “For any $\varepsilon > 0$” after writing “Let $a \in R$ and $\varepsilon > 0$ …” because that means ε represents a particular number. Edward responded, “It doesn’t really matter [meaning the “for any”] because I had $\varepsilon > 0$ there [meaning at the beginning of his proof].” Our interpretation is that Edward didn’t really have a feeling of rightness/appropriateness for this < situation, action > pair, even after considerable experience with it. Furthermore, Edward made similar comments several times thereafter.

4.2 Recognizing a situation, knowing the warrant for the action, carrying out the action, but not having a feeling of rightness/appropriateness

In this instance, a student, Mary, reported that, in the beginning, she carried out the action solely based on her confidence in the professor’s guidance and the belief that following his guidance was necessary to obtain a good grade, rather than because of a feeling of rightness/appropriateness. On reflection, she reported that it was several months later before the action took on a feeling of rightness/appropriateness.

The setting was the first semester of a two-semester beginning graduate real analysis course taught by Dr. K, two years before a retrospective interview with Mary. In addition to the content of the course, Dr. K emphasized writing clear proofs. He required that just a few proofs be handed in every week, but he graded them very meticulously, writing detailed comments on the students' papers. In particular, for theorems starting "For all *x*" he wanted to "fix" the *x*, that is, meaning that *x* should be arbitrary, but fixed within the proof.

4.2.1 The case of Mary

When Mary was asked about this < situation, action > pair she said the following.

> M: At that point [early in the course] my biggest idea was, well he said to "do it," so I'm going to do it because I want to get full credit. And so I didn't have a real sense of why it worked.
>
> I: Did you have any feeling … if it was positive or negative, or extra …
>
> M: Well, I guess I had a feeling of discomfort …
>
> I: Did this particular feature [having to fix *x*] keep coming up in proofs?
>
> M: … it comes up a lot and what happened, and I don't remember when, is that instead of being rote and kind of uncomfortable, it started to just make sense … By the end of the first semester this was very comfortable for me.

At the time of the interview some two years later, Mary said she could think of no other way to write such a proof.

In a subsequent interview with Dr. K, he said in this respect, " … few if any [of the students in his real analysis course] don't have this problem [at the beginning of the course] in my experience." Dr. K recalls having had many conversations with Mary about this particular point. His recommendation to her, and to the other students, was to write at the end of such proofs, something like "As *x* was chosen arbitrarily, we have shown it for all *x*." In a later personal communication, Mary informed us that, for each such proof, she had convinced herself that this final line was justified. By the middle of the first semester, when Mary had become accustomed to "fixing *x*," she was no longer required by Dr. K to write this final sentence.

5. Conclusion

In both the cases of Edward and Mary, a feeling of rightness/appropriateness did not arise more or less simultaneously with the recognition of the < situation, action > pair and understanding the warrant for it. On the contrary, the feeling of rightness/appropriateness did not arise for Mary for considerable time, and we do not have evidence that Edward ever gained a feeling of rightness/appropriateness for this particular action. We think that nonemotional cognitive feelings play an important role, not only in constructing and validating proofs, but also in other areas of mathematical activity where judgment is required.

6. Future research

The first author has informally observed in tutoring a beginning graduate student, from a similar Modified Moore Method course, that sometimes when she is asked to prove a theorem, nothing seems to come to her, despite looking back over the notes. We conjecture that, after awhile, she experiences a feeling of confusion, although she does not give any outward signs of being perturbed. After that, she seems to be “grasping at straws” because she writes almost anything, and we cannot tell “where she is coming from.” It is as if she has abandoned all sense making. We suggest that, when she experiences a feeling of confusion, there is a behavioral schema that causes her to make seemingly random guesses, only very loosely connected to what’s going on in the proof. If our conjecture is correct, this feeling of confusion may become associated with her perception of the situation and influence her behavioral schema to generate inappropriate actions. This phenomenon bears further investigation.

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